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1992 J. Phys. A: Math. Gen. 25 L15

(http://iopscience.iop.org/0305-4470/25/1/003)

J. Phys. A: Math. Gen. 25 (1992) L15-L20. Printed in the UK

LETTER TO THE EDITOR

A note on affine Toda couplings

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Received 24 July 1991

Abstract. We prove the Clebsch–Gordan property for the couplings of an (untwisted) affine Toda field theory utilizing Dorey's conjecture.

The last two years have seen a renewed interest in a family of massive two-dimensional field theories, the affine Toda field theories (ATFT). First investigated [1-3] over a decade ago, these classically completely integrable field theories have reappeared in the context of perturbations of conformal field theories away from criticality [4–7]. The ATFT have many remarkable properties. S-matrices have been proposed [7-13] which satisfy unitarity, crossing and the bootstrap and these agree [1, 9, 12, 14-17] with low order perturbative calculations. The theories are also amenable to thermodynamic Bethe ansatz techniques [5, 18]. Further, many of the classical properties of ATFT appear to survive quantization and this has led to the belief that such theories may be completely integrable at the quantum level. Thus in the simply laced cases it has been found [9, 12] that to low orders in perturbation theory the classical mass ratios are preserved, a necessary ingredient for the S-matrices, and those couplings that vanish classically remain vanishing for on-shell fields [19]. These calculations involve some tremendous cancellations which are due in large part to the very special classical parameters of the theory.

This letter will prove a property of the classical couplings, the Clebsch-Gordan property discussed below, that has been observed [9, 11, 15, 16, 18] in these theories and verified to persist at low orders of perturbation theory. To describe this more precisely we first recall some of the classical properties of ATFT.

The (untwisted) affine Toda lagrangian is

$$\mathscr{L}=\frac{1}{2}(\partial_\mu\phi,\partial^\mu\phi)-\frac{m^2}{\beta^2}\sum_{\alpha\in\bar\Delta} n_\alpha\, \mathrm{e}^{\beta(\alpha,\phi)}. \tag{1}$$

Here $\bar\Delta=\Delta\cup\{-\Theta\}$ is the set of simple roots together with minus the highest root $\Theta=\Sigma_{\alpha\in\Delta} n_\alpha\alpha$ of a simple Lie algebra g of rank n. We adopt the convention $n_{-\Theta}=1$ so the Coxeter number h of g becomes $h=\Sigma_{\alpha\in\bar\Delta} n_\alpha$. Let λ_α denote the fundamental weights of g i.e. if $\beta\in\Delta$ then $(\lambda_\alpha,\beta^\vee)=\delta_{\alpha,\beta}$ (where $\alpha^\vee=2\alpha/(\alpha,\alpha)$) and denote the Cartan matrix by $K_{\alpha\beta}=(\alpha^\vee,\beta)\alpha,\beta\in\Delta$. By expanding the potential term in (1) we get

$$V(\phi)=\frac{m^2h}{\beta^2}+\frac{1}{2}\phi^iM^2_{ij}\phi^j+\frac{1}{3!}c_{ijk}\phi^i\phi^j\phi^k+\dots \tag{2}$$

Here $M^2=m^2\Sigma_{\alpha\in\bar\Delta} n_\alpha\alpha\otimes\alpha$ and $c=m^2\beta\,\Sigma_{\alpha\in\bar\Delta} n_\alpha\alpha\otimes\alpha\otimes\alpha$ are the mass matrix and three point couplings. The notation is such that if $\phi=\phi^ie_i$ is a field in a chosen basis

$\{e_i\}$ then c_{ijk} is the three point coupling in this basis: $c_{ijk} = c(e_i, e_j, e_k) = m^2\beta \sum_{\alpha\in\tilde{\Delta}} n_\alpha(\alpha, e_i)(\alpha, e_j)(\alpha, e_k)$. We assume that the classical mass matrix has been diagonalized; in the quantum theory the basis will be defined order by order in perturbation theory. The highly constrained nature of these Lagrangians is manifested in the classical mass spectrum [3] and three point couplings.

(1) If we form the vector of masses, $\boldsymbol{m} = (m_1, m_2, \ldots, m_n)$, the mass spectrum of the (untwisted) affine Toda theories is succinctly expressed [9, 11] by: the *vector* $\boldsymbol{m}$ is the (left) *Perron–Frobenius eigenvector of the Cartan matrix* of g. This eigenvector† is the only eigenvector of K whose entries are all of the same sign; we have $K^{\mathrm{T}}\boldsymbol{m} = 4\sin^2(\pi/2h)\boldsymbol{m}$. The correspondence enables one to identify the particle of mass m_i with the fundamental representation λ_i of g. The higher conserved currents of (1) are connected with the remaining eigenvectors of the Cartan matrix [18, 20]. Unfortunately this correspondence fails for the twisted simply laced algebras. A point of notation: when g has complex representations it is helpful to use a complex basis for the equal mass fields; in this case $\bar{i}$ denotes the conjugate field to particle i. Perron–Frobenius (PF) eigenvectors arise in the construction of solvable statistical mechanical models [21] and the study of operator algebras [22]; they were also observed [18] in the context of elastic S-matrices independent of any underlying field theory. The ratios of the masses are very special algebraic numbers. For example, if the lowest mass is scaled to be 1 then products $m_i m_j$ are in fact positive integral combinations of the masses (when $g \neq d_{\mathrm{odd}}, e_7, b_{\mathrm{even}}$ in which case the domain of the coefficients must be enlarged)‡.

(2) The non-zero three point couplings of the simply-laced algebras obey [9, 7, 10] the 'area rule'

$$c_{ijk} = \varepsilon_{ijk}\frac{4\beta}{\sqrt{h}}\Delta_{ijk} \tag{3}$$

where $\varepsilon_{ijk} = \pm 1$ and Δ_{ijk} is the area of the triangle with sides m_i, m_j and m_k. There are only slight changes to this formula for the non-simply laced cases: the twisted algebras have h replaced by g while the untwisted non-simply laced algebras obey (3) unless each of i, j, k correspond to short roots. In this latter case the coupling is reduced [7, 12, 9, 10] by the ratio $(\nu-1)\sqrt{(\Theta_S, \Theta_S)/(\Theta_L, \Theta_L)}$, where ν is the order of the simply laced Dynkin diagram automorphism that upon folding leads to g and $\Theta_{L(S)}$ is the highest long (short) root. (Because the untwisted b_n algebra has no couplings between three short vectors (3) is unmodified in this case.) In terms of the fusion angle θ_{ij}^k, $\Delta_{ijk} = \frac{1}{2}m_i m_j \sin\theta_{ij}^k$. These angles are all integer multiples of π/h. Using the correspondence between masses and representations enabled by the PF eigenvector one observes a necessary (though in general not sufficient) condition for c_{ijk} to be non-zero is that the irreducible decomposition of the tensor product of representations $V(\lambda_i)\otimes V(\lambda_j)\otimes V(\lambda_k)$ should contain the trivial representation. Here $V(\lambda)$ is the irreducible representation with highest weight λ. This is known [11, 15, 16] as the Clebsch-Gordan (CG) property of ATFT.

† Our definition of the Cartan matrix dictates that we specify the left eigenvector of K i.e. the right eigenvector of K^{T}. This distinction is irrelevant for the simply laced algebras. We note that for g_2 and f_4 the left and right Perron–Frobenius eigenvectors have the same entries but in reversed orders: that we choose the particular eigenvector stated follows from either the Clebsch-Gordan property described below, or from folding. The left eigenvector of b_n is of course the right eigenvector of c_n.

‡ Werner Nahm [23] suggests this property is related to generalized Penrose tilings. The listed simply laced exceptions are also distinguished in Ocneanu's work [22].

(3) Further one can show for the simply laced (and even the twisted non-simply laced) cases each particle has $h-2$ non-vanishing three point couplings. More precisely for fixed k there are $h-2$ ordered pairs (i,j) for which c_{ijk} is non-zero. In the untwisted non-simply laced case this is modified so that if k corresponds to a long root there are $g-2$ ordered pairs while if k corresponds to a short root there are $h-(\Theta_L,\Theta_L)/(\Theta_S,\Theta_S)$ ordered pairs.

We have not as yet specified precisely which couplings are non-vanishing. Using the PF correspondence this is embodied in: a non-zero coupling c_{ijk} exists if and only if there exist integers r and s such that

$$\lambda_i + c^r\lambda_j + c^s\lambda_k = 0 \tag{4}$$

and c is a Coxeter element. Here (4) is an equivalent form of Dorey's conjecture[20] to which we now turn. Having established (4) we then prove as a consequence the CG property, showing why this is not a sufficient condition to determine the non-zero couplings.

Because the Dynkin diagram of g is a tree it is a bipartite graph; consequently the simple roots $\Delta=\{\alpha_1,\dots\alpha_n\}$ may be partitioned into two sets of mutually orthogonal roots: $\Delta=\circ\cup\bullet$, where $\circ=\{\alpha_1,\dots\alpha_k\}$ and $\bullet=\{\alpha_{k+1},\dots\alpha_n\}$. A Coxeter element c is then a product of all of the simple reflections w_α defined by the simple roots $\alpha\in\Delta$. Different orderings of the simple roots lead to conjugate Coxeter elements and for our purposes it is useful to write $c=c_\circ c_\bullet$ where $c_\circ=\Pi_{\alpha\in\circ}\, w_\alpha$ and similarly for $c_\bullet$. In terms of their actions on the roots we have

$$c_\circ=\begin{pmatrix}-1 & -K^{\mathrm{T}}_{\circ\bullet}\\ 0 & 1\end{pmatrix} \qquad c_\bullet=\begin{pmatrix}1 & 0\\ -K^{\mathrm{T}}_{\bullet\circ} & -1\end{pmatrix}. \tag{5}$$

Steinberg [24] and Kostant [25] have established many beautiful results describing the action of Coexter elements c on Φ, the root system of g. Of relevance for ATFT is the fact c separates Φ into n orbits: $\Phi=\bigcup_{i=1}^n\Phi_i$ where $\Phi_i=\{c^r\phi_i\colon 0\leqslant r\leqslant h-1\}$ and the ϕ_i are the positive roots of Φ which become negative roots under the action of c. Explicitly† $\phi_i=w_{\alpha_n}w_{\alpha_{n-1}}\dots w_{\alpha_{i+1}}\alpha_i$ whence $\phi_i=c_\bullet\alpha_i$ $i=1,\dots k$ and $\phi_j=\alpha_j$ $j=k+1,\dots n$. We see $c\phi_i=-\alpha_i$ $i=1,\dots k$ and $c\phi_j=-c_\circ\alpha_j$ $j=k+1,\dots n$ thus $\varepsilon(i)\alpha_i\in\Phi_i$ where $\varepsilon(i)=1$ $(\alpha_i\in\bullet)$ and $\varepsilon(i)=-1$ $(\alpha_i\in\circ)$ are 'colour' factors. Further, those elements in each orbit which are positive roots are nicely characterized: $\Phi^+\cap\Phi_i=\{c^{-r}\phi_i\colon 0\leqslant r\leqslant(h/2)-1\}$ when h is even or when h is odd (only the case a_{even}) $\Phi^+\cap\Phi_i=\{c^{-r}\phi_i\colon 0\leqslant r\leqslant H\}$ where $H=(h-1)/2-1$ if $i\leqslant k$ or $H=(h-1)/2$ if $k+1\leqslant i$. Equivalently the longest word $w_\circ$ of the Weyl group is $c^{h/2}$ when h is even and $w_\bullet c^{(h-1)/2}$ when h is odd. These facts follow from a connection‡ between the eigenvalues and eigenvectors of the Cartan matrix and Coxeter element [28] which is the reason [30, 31] behind the PF correspondence of ATFT masses. This connection is made manifest with the identity of Steinberg [26]:

$$c_\circ + c_\bullet = 2 - K^{\mathrm{T}} \tag{6}$$

which follows from (5). To each eigenvector of the Cartan matrix can one associate a two-plane in root space on which the Coxeter element acts naturally. Thus if x is a (right) eigenvector with corresponding eigenvalue $\hat{x}$ then upon setting $\mu=\Sigma_\circ\, x_i\lambda_i$ and

† Steinberg [24] deals with the roots $\rho_i=-c\phi_i$.

‡ Indeed these connections extend to the affine root systems as well [26–28], though some changes are needed [29] for the case of $\bar{a}_{\text{even}}$.

$\nu = \Sigma_\bullet x_i \lambda_i$ one has $\Sigma_\Delta x_i \alpha_i = \hat{x}(\mu + \nu)$ and $\Sigma_\bullet x_i \alpha_i = (\hat{x} - 2)\mu + 2\nu$. These suffice to show the plane spanned by μ, ν is left invariant by the dihedral group $D_h = \langle c_\circ^2 = c_\bullet^2 = c^h = 1 \rangle$. In particular when $\boldsymbol{x} = \boldsymbol{m}$, the PF eigenvector, each root of Φ_i projects onto a circle of radius m_i on this plane and D_h acts faithfully. In this case the lines defined by μ and ν are at an angle of π/h and $\alpha \in \circ$ projects onto μ while $\alpha \in \bullet$ projects onto ν.

With this background Dorey's conjecture [20] takes the following form. There is a non-zero coupling between three particles i, j, k if and only if there exist integers r, s such that

$$\phi_i + c^r \phi_j + c^s \phi_k = 0. \tag{7}$$

Actually Dorey's conjecture is made in a context somewhat wider [32] than ATFT but has only been proven in the more restricted situation of (untwisted) affine Toda theories. We initially verified (7) on a case by case basis simply enumerating the possibilities for exceptional g; more recently an elegant invariant proof of both this and (3) has been given [31] using Lie algebraic techniques. Many of the classical results described so far can be proven on the basis of (7) and Steinberg's results: because of the projection properties onto the plane associated with the Perron–Frobenius eigenvector the resulting triangles must have angles with integer multiples of π/h and (3) holds; in the simply laced case Dorey [20] has argued that the $h-2$ property also results. A caution is perhaps warranted at this juncture: the pair (r, s) arising from (7) is not unique; there are in fact two such pairs for each such coupling specified in this form. That there are at least two arises because we can always 'flip' the triangle (replace c with c^{-1} in our discussion); that there are only two follows from elementary geometry or by examining the allowed ranges of r and s. For example Steinberg's characterization of the positive roots forbids $1 - [h/2] \leq r, s \leq 0$ as the sum of three positive roots cannot vanish, and so on. If (r, s) and (r', s') are the two flipped pairs then $r + r' + \frac{1}{2}[\varepsilon(j) - \varepsilon(i)] = h$.

It remains to demonstrate the CG property follows from (7) and for this it easier to first show its equivalence with (4). We may express the fundamental weights λ_i in terms of the ϕ_i as follows [25]:

$$\phi = \lambda_i - c^{-1}\lambda_i. \tag{8}$$

(Using $(1-c)^{-1} = -1/h\, \Sigma_{p=1}^{h-1} pc^p = 1/h\, \Sigma_{p=1}^{h} pc^{-p}$ this may be inverted to yield $\lambda_i = 1/h\, \Sigma_{p=1}^{h} pc^p \phi_i$; the averaging operators, $P_s = 1/h\, \Sigma_{k=0}^{h-1} \omega^{-sk} c^k$, where ω is an hth root of unity, are projectors.) Using (8) we may express (7) in the following form

$$\lambda_i + c^r \lambda_j + c^s \lambda_k = c(\lambda_i + c^r \lambda_j + c^s \lambda_k).$$

Now as the Coxeter element has no fixed nonzero eigenvector this equation requires both sides to vanish. Thus we arrive at our characterization of the coupling (4).

Finally we show that $V(\lambda_{\bar{i}})$ is an irreducible component of $V(\lambda_j) \otimes V(\lambda_k)$ so proving the CG property. (Our notation was such that $V(\lambda_{\bar{i}}) = V(\lambda_i)^*$.) To prove this we recall a conjecture of Parthasarathy, Ranga Rao and Varadarajan [33] (PRV) only recently proven [34]. The PRV conjecture says that for any element σ in the Weyl group W of g and any highest weights λ and μ then $V([\lambda + \sigma\mu])$ occurs with multiplicity at least one in $V(\lambda) \otimes V(\mu)$, where $[\lambda + \sigma\mu]$ is the dominant weight conjugate to $\lambda + \sigma\mu$. Actually Kostant gave a strengthened form of the PRV conjecture which gives the multiplicities but we shall not need this here.

Combining the PRV result and (4) we have $V([-c^{r-s}\lambda_i])$ is an irreducible component of $V(\lambda_j) \otimes V(\lambda_k)$. Now $[-c^{r-s}\lambda_i] = [-\lambda_i]$ and to obtain the CG property we observe

$[-\lambda_i] = \lambda_{\bar{i}}$. Indeed $w_0\lambda_i = -\lambda_{\bar{i}}$. (When g has only real representations $w_0 = c^{h/2}$ is the central inversion which sends all roots to their negatives.)

Thus the CG property is a direct consequence of (4) and we have shown ATFT possess this at the classical level. Before concluding however it is instructive to illustrate how the 'holes' in the CG-coupling correspondence arise. These are the situations when the CG decomposition allows a coupling that actually vanishes in the Toda theory. A simple example is given [16, 11] by d_5. With λ_2 denoting the adjoint representation we have $V(\lambda_2) \subset V(\lambda_2) \otimes V(\lambda_2)$ but there is no corresponding coupling c_{222}. In this case one finds $\lambda_2 + c^3\lambda_2 = \sigma\lambda_2$ where $\sigma = w_{\alpha_4} w_{\alpha_5} w_{\alpha_3} w_{\alpha_2}$. The point is that $-\sigma$ is not conjugate to a power of c and so (4) fails to be satisfied. The PRV result, which allows conjugation by an arbitrary element of the Weyl group, is less stringent and that is why more non-zero CG allowed couplings exist than actually arise in Toda field theory. The ATFT couplings are even more restricted than g invariance would dictate: one suspects this hidden symmetry is part of the reason behind their striking classical and quantum behaviour.

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